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OPTIMAL DESIGN OF VIBRATING RECTANGULAR PLATES

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Abstract—The thickness function of a simply supported rectangular plate is determined such that the fundamental natural frequency of transverse vibrations attains an optimum value. The volume of the solid plate, its size, and data pertaining to its material properties, are assumed to be given.

The problem consists of determining the deflection function corresponding to the optimal plate thickness function from a non-linear, fourth order partial differential eigenvalue problem, derived by variational analysis.

Second and higher order normal derivatives of the deflection function are singular along the boundary of the domain. This behaviour is investigated analytically, and taken into account in a finite difference formulation of the problem, which is solved numerically by successive iterations.

INTRODUCTION

In the present paper, a variational method is applied for the solution of an optimization problem, governed by *partial* differential equations.

The method is similar to the one developed by Niordson[1] in 1965, and later applied by Keller and Niordson[2], by the author[3], and by Karihaloo and Niordson[4, 5]. However the problems dealt with in these papers are all related in the sense that they can be reduced to a description by one independent variable, i.e. they result in *ordinary* differential equations.

The present work deals with the two-dimensional problem of optimizing a thin, simply supported, rectangular solid plate, the volume, length and width, and Poisson's ratio of which are assumed to be given. The thickness function is determined from the condition that the fundamental natural frequency of transverse vibrations attains a maximum value.

The analysis is based upon the general theory of *thin* elastic plates, and the starting point is the linear, fourth order partial differential equation of free, harmonic, transverse vibrations of a plate with an arbitrarily non-uniform thickness function. In connection with the boundary conditions, this differential equation constitutes a fully-definite and self-adjoint eigenvalue problem.

Treating the Rayleigh quotient expression of this problem by the variational approach mentioned above, we derive a highly non-linear, fourth order partial differential equation coupled with a partial integral equation. This constitutes a fundamental eigenvalue problem for the optimal deflection function, and a parameter.

This paper is also concerned with the solution of this eigenvalue problem, which in fact constitutes a major part of the treatise. For this, a procedure of successive iterations is developed and implemented by a finite difference technique.

Second and higher order normal derivatives of the deflection function are singular along the simply supported boundaries of the rectangular plate. Therefore, it is necessary to determine the types of these singularities analytically and to pay considerable attention to them when developing the finite difference procedure.

The problem is found to be complicated by the fact that the construction of a formal integration procedure ensuring convergence by successive iterations is not immediately obvious. In fact, the solution cannot be based entirely upon integrations; some numerical differentiations are unavoidable.

Numerical solutions of the optimization problem are obtained for different values of the plate aspect ratio. As an asymptotic case, we find the optimal solution of a simply supported vibrating *beam* of non-uniform height and constant width when the aspect ratio tends to infinity.

The plate solutions obtained have rather smoothly varying thickness functions and must be considered as local optima when compared with corresponding integrally stiffened plate structures. It will finally be shown that a global optimum solution does not exist for the problem under study.

TRANSVERSE VIBRATIONS OF PLATES OF NON-UNIFORM THICKNESS

In the theory of thin elastic plates of non-uniform thickness, the deflection function w corresponding to free, harmonic, transverse vibrations is governed by the dimensionless partial differential equation

$$\{h^3[(1-\nu)w_{,\alpha\beta} + \nu\delta_{\alpha\beta}w_{,\gamma\gamma}]\}_{,\alpha\beta} = \lambda hw, \tag{1}$$

where the parameter λ is given by the expression

$$\lambda = \frac{\omega^2\rho a^8 12(1-\nu^2)}{EV^2}. \tag{2}$$

The equation refers to a set of dimensionless Cartesian coordinates x_α ($\alpha = 1, 2$) given by

$$x_\alpha = X_\alpha/a, \tag{3}$$

where X_α are the original coordinates, and a is some characteristic length of the plate. The commas in (1) indicate partial differentiation with respect to succeeding indices, and repeated greek indices imply summation. Furthermore, $\delta_{\alpha\beta}$ is the Kronecker delta, and $h(x_\alpha)$ denotes a dimensionless plate thickness function, defined by

$$h = Ha^2/V, \tag{4}$$

where $H(x_\alpha)$ is the actual thickness function and V is the volume of the plate.

In equation (2), ω is the natural angular frequency of the plate, and ρ, ν and E denote the mass density, Poisson ratio and Young's modulus, respectively, of the plate material.

Now, let the x_1-axis coincide with a short side, and the x_2-axis with a long side of our rectangular plate, and let us choose the characteristic length a equal to that of a short side. Hereby, as Fig. 1 indicates, our domain is bounded by the coordinate axes and by the lines $x_1 = 1$ and $x_2 = c$, where $c \geq 1$ is the aspect ratio of the plate.

As is well known, the boundary conditions for a simply supported plate can be formulated as

$$w = 0 \tag{5}$$

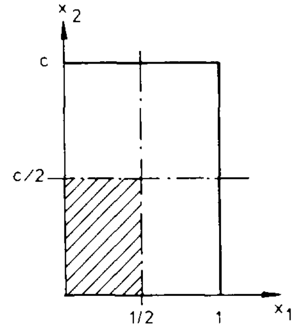

Fig. 1. The plate domain.

and

$$h^3\,\Delta w = 0 \tag{6}$$

along the boundary contour just mentioned. In (6), Δ denotes the Laplacian operator.

The differential equation (1) and the boundary conditions (5) and (6) constitute a linear eigenvalue problem. The eigenvalues λ and corresponding eigenfunctions $w(x_\alpha)$ of this problem are all dependent on, and uniquely determined by, the thickness function $h(x_\alpha)$.

By applying the divergence theorem and by considering the boundary conditions, we see that the eigenvalue problem is self-adjoint and fully-definite. Furthermore, the investigation of the fully-definiteness reveals the following expression for the Rayleigh quotient of the problem,

$$R[w] = \frac{\int_0^1 \int_0^c h^3\{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}\, w_{,22}]\}\, dx_1\, dx_2}{\int_0^1 \int_0^c h w^2\, dx_1\, dx_2} = \lambda. \tag{7}$$

It is well known that the Rayleigh quotient (7) equals an eigenvalue λ of the problem (1), (5) and (6) if the corresponding eigenfunction $w(x_\alpha)$ is inserted. This is also expressed in (7).

VARIATIONAL ANALYSIS

Let us from now on assume the volume V of the plate to be given. According to (3) and (4), we hereby impose the following constraint upon the dimensionless thickness function $h(x_\alpha)$:

$$\int_0^1 \int_0^c h\, dx_1\, dx_2 = 1. \tag{8}$$

Furthermore, let λ denote the fundamental eigenvalue, and w the corresponding eigenfunction.

In order to find out whether it is possible to determine a thickness function h, obeying (8) and rendering λ an optimal value, we shall now generalize the variational procedure first outlined in[1] into the field of continua with two independent spatial variables.

Assuming an optimal thickness function to exist, we consider a family of non-negative functions $h(x_\alpha, \varepsilon)$, containing the optimal solution $h(x_\alpha, 0)$. The members of this family are assumed to satisfy the volume constraint (8) and to depend differentiably upon the parameter ε.

Associated with every thickness function $h(x_\alpha, \varepsilon)$, is a first eigenfunction $w(x_\alpha, \varepsilon)$. Due to a well known theorem for differential equations, the functions $w(x_\alpha, \varepsilon)$ will be differentiable with respect to ε, since they are solutions of eigenvalue problems with the functions $h(x_\alpha, \varepsilon)$. Furthermore, any $w(x_\alpha, \varepsilon)$ is an *admissible* function for the eigenvalue problem related to the optimal thickness function $h(x_\alpha, 0)$, since it is sufficiently differentiable with respect to x_α and satisfies the kinematical boundary conditions.

Let us now define a functional R, depending upon the parameters ε and δ only:

$$R[\varepsilon, \delta] = \frac{\int_0^1 \int_0^c h(x_\alpha, \varepsilon)^3 U(x_\alpha, \delta)\, \mathrm{d}x_1\, \mathrm{d}x_2}{\int_0^1 \int_0^c h(x_\alpha, \varepsilon) w(x_\alpha, \delta)^2\, \mathrm{d}x_1\, \mathrm{d}x^2} \tag{9}$$

where

$$U(x_\alpha, \delta) = [\Delta w(x_\alpha, \delta)]^2 + 2(1 - \nu)[[w_{,12}(x_\alpha, \delta)]^2 - w_{,11}(x_\alpha, \delta) w_{,22}(x_\alpha, \delta)].$$

This functional represents the Rayleigh quotient for the eigenvalue problem corresponding to the thickness function $h(x_\alpha, \varepsilon)$, and it is identical with the eigen value $\lambda(\varepsilon)$ of this problem for $\varepsilon = \delta$. Specifically, for $\varepsilon = \delta = 0$, we have $R[0, 0] = \lambda(0)$, the optimal value to be determined. A *necessary* condition for this is expressed by the equation

$$(\mathrm{d}\lambda/\mathrm{d}\varepsilon)_{\varepsilon=\delta=0} = 0 \tag{10}$$

or

$$(\mathrm{d}R/\mathrm{d}\varepsilon)_{\varepsilon=\delta=0} = (R_{,\varepsilon})_{\varepsilon=\delta=0} + (R_{,\delta})_{\varepsilon=\delta=0}(\mathrm{d}\delta/\mathrm{d}\varepsilon)_{\varepsilon=\delta=0}\,. \tag{11}$$

In fact, $R[0, \delta]$ represents the Rayleigh quotient related to the optimal thickness function $h(x_\alpha, 0)$.

Of all admissible functions $w(x_\alpha, \delta)$, this quotient will be minimized by the first eigenfunction $w(x_\alpha, 0)$ corresponding to $h(x_\alpha, 0)$, in accordance with Rayleigh's minimum principle. Thus we have $(R_{,\delta})_{\varepsilon=\delta=0} = 0$, so that (11) reduces to

$$(R_{,\varepsilon})_{\varepsilon=\delta=0} = 0. \tag{12}$$

Applying this condition to (9), dividing through by the denominator, and taking (7) into account, we arrive at the equation

$$\int_0^1 \int_0^c [3h^2\{(\Delta w)^2 + 2(1 - \nu)[(w_{,12})^2 - w_{,11} w_{,22}]\} - \lambda w^2](h_{,\varepsilon})_{\varepsilon=0}\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0, \tag{13}$$

where the optimal thickness function, the corresponding deflection function and the optimal eigenvalue are simply denoted by h, w and λ, respectively.

From the volume constraint (8), which also has to be satisfied by the optimal thickness function, we obtain

$$\int_0^1 \int_0^c (h_{,\varepsilon})_{\varepsilon=0}\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0. \tag{14}$$

Equations (13) and (14) are both satisfied if the multiplying expression of $(h_{,\varepsilon})_{\varepsilon=0}$ in the integrand of (13) is independent of x_α, i.e. constant. Thus we arrive at the *optimality condition* of the present problem:

$$3h^2\{(\Delta w)^2 + 2(1 - \nu)[(w_{,12})^2 - w_{,11} w_{,22}]\} - \lambda w^2 = \lambda A. \tag{15}$$

Subsequently, a determining equation for the constant A will be derived. However, let us first solve (15) for the possible optimal plate thickness function h

$$h(x_\alpha) = \left(\frac{\lambda}{3}\right)^{1/2} \left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{1/2} \tag{16}$$

and find a convenient explicit equation for the corresponding optimal eigenvalue λ by combining (16) and (8):

$$\lambda = \frac{3}{\left(\int_0^1 \int_0^c \left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{1/2} \mathrm{d}x_1\, \mathrm{d}x_2\right)^2}. \tag{17}$$

If we now multiply (15) by h, integrate over the plate domain, and combine equations (7), (8), (16) and (17), we get the following *implicit* equation for the quantity A in terms of $w(x_\alpha)$,

$$A = 2\,\frac{\int_0^1 \int_0^c \left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{1/2} w^2\, dx_1\, \mathrm{d}x_2}{\int_0^1 \int_0^c \left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{1/2} \mathrm{d}x_1\, \mathrm{d}x_2}. \tag{18}$$

In order to obtain an equation for the optimal deflection function w, we substitute (16) in the differential equation (1) and get

$$\left\{\left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{3/2} [(1-\nu)w_{,\alpha\beta} + \nu\delta_{\alpha\beta}w_{,\gamma\gamma}]\right\}_{,\alpha\beta}$$
$$= 3\left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{1/2} w. \tag{19}$$

The boundary conditions for w become, cf. (5), (6) and (16),

$$w = 0 \tag{20}$$

and

$$\left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{3/2} \Delta w = 0 \tag{21}$$

invoked along the lines

$$(x_1, x_2) = \begin{cases} (0, x_2) & \\ (1, x_2) & 0 \le x_1 \le 1 \\ (x_1, 0) & 0 \le x_2 \le c. \\ (x_1, c) & \end{cases}$$

By studying the non-linear partial differential equations obtained by the variational procedure, we note firstly two equations, (16) and (17), explicitly expressing the optimal thickness function h and the optimal value of the first natural frequency λ, respectively, in terms of the corresponding deflection function w and a parameter A.

Secondly, we have obtained a highly non-linear, fourth order partial differential equation (19), coupled with a partial integral equation (18). These two equations, in connection with the boundary conditions (20) and (21), constitute a fundamental eigenvalue problem for the deflection and the parameter A.

If $w(x_\alpha)$ and the corresponding value of A are known, $h(x_\alpha)$ and λ can easily be computed by substitution in (16) and (17). Therefore, the solution of the non-linear, partial differential eigenvalue problem (18)–(21) is the principal object of the procedure.

The parameter A can be interpreted as the eigenvalue of this non-linear eigenvalue problem and, furthermore, it can be simply correlated with the optimal frequency λ: Observe that the problem (18)–(21) is homogeneous and that it is solved by $Kw(x_\alpha)$ and K^2A, K being any finite constant, provided the function $w(x_\alpha)$ and the corresponding A designate a solution. Hence we are free to normalize the problem, for example by extending the set of equations by the following equation

$$\int_0^1 \int_0^c \left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{1/2} w^2 \, dx_1 \, dx_2 = \frac{\sqrt{3}}{2}. \tag{22}$$

Normalizing by prescribing this value of the right-hand numerator in (18), we identify A as $\lambda^{1/2}$.

A closed form solution of the non-linear eigenvalue problem stated above is not obtainable. Therefore it will be solved numerically by a successive iteration procedure based on a finite difference formulation.

Before we proceed to a more detailed description of this procedure, let us for the sake of expediency assume both the optimal plate and its corresponding fundamental deflection function to be symmetrical with respect to the planes $x_1 = 1/2$ and $x_2 = c/2$, see Fig. 1. This enables us to restrict numerical calculations to the smaller domain bounded by the lines $x_1 = 0$, $x_2 = 0$, $x_1 = 1/2$ and $x_2 = c/2$.

Also, it is convenient to compress the equations written above in some detail. Let us do this by introducing the shorter notation D for the function

$$D(x_\alpha) = \left[\frac{A + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{3/2}, \tag{23}$$

which is proportional to the optimal plate thickness function cubed, cf. (16), and hence proportional to the optimal plate bending stiffness function.

Now, in terms of D, the expressions of λ and h become

$$\lambda = \frac{3}{16\left(\int_0^{1/2} \int_0^{c/2} D^{1/3} \, dx_1 \, dx_2\right)^2} \tag{24}$$

and

$$h(x_\alpha) = \left(\frac{\lambda}{3}\right)^{1/2} D^{1/3}. \tag{25}$$

The fourth order, non-linear partial differential equation (19) and the integral equation (18) can be written in shorter form as

$$\{D[(1-\nu)w_{,\alpha\beta} + \nu\delta_{\alpha\beta} w_{,\gamma\gamma}]\}_{,\alpha\beta} = 3D^{1/3}w \tag{26}$$

and

$$\Lambda = 2\frac{\int_0^{1/2}\int_0^{c/2} D^{1/3}w^2\,\mathrm{d}x_1\,\mathrm{d}x_2}{\int_0^{1/2}\int_0^{c/2} D^{1/3}\,\mathrm{d}x_1\,\mathrm{d}x_2}, \tag{27}$$

respectively. Recall that (27) is an implicit equation for the (positive) eigenvalue Λ.

Due to the assumed symmetry of w, the kinematical boundary conditions (20) can be replaced by the conditions

$$w(x_1, 0) = w(0, x_2) = w_{,1}(1/2, x_2) = w_{,2}(x_1, c/2) = 0 \tag{28}$$

for the boundary just introduced.

The remaining boundary conditions for the differential equation (26) are still most conveniently expressed in terms of the function on the left-hand side of equation (21). Let us introduce the short notation M for this *scalar* moment function,

$$M(x_\alpha) = \left[\frac{\Lambda + w^2}{(\Delta w)^2 + 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]}\right]^{3/2} \Delta w = D\Delta w \tag{29}$$

which, apart from the multiplying factor $1 + \nu$, is equal to the sum of the moment tensor components M_{11} and M_{22} in the optimal plate.

Bearing the equilibrium conditions of a plate element in mind, we easily see that the bending moments M_{11}, M_{22} and hence the function $M(x_\alpha)$ must be at least once continuously differentiable, since the load of the plate is due to distributed inertia forces only. In combination with the symmetry assumptions† made, this implies that $M_{,1}$ and $M_{,2}$ vanish along the lines of symmetry $x_1 = 1/2$ and $x_2 = c/2$, respectively.

Hence, we can replace the boundary conditions (21) by

$$M(x_1, 0) = M(0, x_2) = M_{,1}(1/2, x_2) = M_{,2}(x_1, c/2) = 0. \tag{30}$$

BEHAVIOUR OF THE SOLUTION NEAR THE BOUNDARY

The solution of the governing eigenvalue problem, (23) and (26)–(30), is singular along the simply supported boundaries $x_1 = 0$ and $x_2 = 0$.

Specifically, this is easily shown at the point $(x_1, x_2) = (0, c/2)$, see Fig. 1. Here, the boundary condition $M = 0$, with M given by (29), reduces to $(w_{,11})^{-2} = 0$, since we have $w_{,22} = 0$ and $w_{,12} = 0$ (due to the symmetry of w). According to equation (25), where D is defined by (23), the revealed infinity of $w_{,11}$ implies a vanishing of the plate thickness at the boundary point considered.

Singularities should be expected[6] unless constraints are imposed on the thickness function[7].

In fact, the singular behaviour strongly influences the numerical solution procedure, and a careful analytical investigation of the singularities is necessary, not just to get a satisfactory numerical representation of the solution, but to obtain a solution at all.

Let us determine the type of the singularity at the boundary $x_1 = 0$, $0 < x_2 \leq c/2$ by assuming the solution to be expandable in a series

$$w(x_\alpha) = a_1(x_2)x_1 + a_2(x_2)x_1^{\,2} + \cdots + g_1(x_2)x_1^{\,p} + \cdots \tag{31}$$

† Observe that D and Δw may both have discontinuous first derivatives normal to the lines of symmetry.

in the vicinity of $x_1 = 0$. All the coefficients are assumed to be sufficiently differentiable functions, depending only upon x_2. Furthermore, p is assumed to be the lowest non-integer power of x_1, corresponding to a non-vanishing coefficient g_1. As the expansion (31) must agree with the boundary conditions $w(0, x_2) = 0$ and $M(0, x_2) = 0$, where M is given by (29), we find, *a priori*, that p must belong to the interval $1 < p < 2$.

Now, by substituting (31) in the differential equation (26), taking (23) into account, and setting the sum of the coefficients of the lowest order terms in x_1 equal to zero, we find p equal to 3/2. From (26), only the terms $Dw_{,1111}$, $D_{,1}w_{,111}$ and $D_{,11}w_{,11}$ contribute to the determining equation for p.

The result, $p = 3/2$, implies that $w_{,11}$ and the optimal plate thickness function h (like $D^{1/3}$) are proportional to $x_1^{-1/2}$ and $x_1^{1/2}$, respectively, for small values of x_1, and this is valid along *any* normal $x_2 = \text{constant}$ $(0 < x_2 \leq c/2)$ to the simply supported boundary line $x_1 = 0$. Similar analytical behaviour was found near the boundary of an optimally designed, simply supported *circular* plate, performing transverse vibrations[3].

However, to obtain the solution numerically in the present case, more detailed knowledge of its analytical behaviour near the simply supported boundaries is necessary, i.e. some following powers in the series expansion (31) must be determined. Doing this successively in a manner similar to that described above, we arrive at

$$w(x_\alpha) = a_1(x_2)x_1 + g_1(x_2)x_1^{3/2} + g_2(x_2)x_1^{5/2} + a_3(x_2)x_1^3 + \dots \tag{32}$$

for small values of x_1, and for $0 < x_2 \leq c/2$. To obtain this expansion, not only the above mentioned terms in (26), but also the terms $D_{,11}w_{,22}$, $D_{,1}w_{,122}$ and $D_{,12}w_{,12}$ must be encountered in the analysis.

Let us now use (32) to compute the following series expansions for Δw, D, $D^{1/3}$ and M, the key functions of the problem. We find

$$\Delta w = \tfrac{3}{4} g_1 x_1^{-1/2} + \tfrac{15}{4} g_2 x_1^{1/2} + (6a_3 + a_{1,22})x_1 + \dots, \tag{33}$$

$$D = \frac{64}{27}\frac{A^{3/2}}{|g_1^3|}\left[x_1^{3/2} - \frac{8}{3}\frac{k_1}{g_1^2}x_1^{5/2} - \frac{8}{3}\frac{k_2}{g_1^2}x_1^3 + \dots\right], \tag{34}$$

$$D^{1/3} = \frac{4}{3}\frac{A^{1/2}}{|g_1|}\left[x_1^{1/2} - \frac{8}{9}\frac{k_1}{g_1^2}x_1^{3/2} - \frac{8}{9}\frac{k_2}{g_1^2}x_1^2 + \dots\right], \tag{35}$$

and

$$M = \frac{64}{27}\frac{A^{3/2}}{|g_1^3|}\left[\frac{3}{4}g_1 x_1 + k_3 x^{5/2} + \dots\right], \tag{36}$$

in the vicinity of $x_1 = 0$, $0 < x_2 \leq c/2$. The x_2-dependent functions k_1, k_2 and k_3 are given by

$$k_1(x_2) = \tfrac{45}{8} g_1 g_2 + 2(1 - \nu)(a_{1,2})^2$$

$$k_2(x_2) = 9g_1 a_3 + 6(1 - \nu)g_{1,2}a_{1,2} + \tfrac{3}{2}\nu g_1 a_{1,22}$$

$$k_3(x_2) = -12a_3 + (1 - 3\nu)a_{1,22} - 12(1 - \nu)\frac{g_{1,2}a_{1,2}}{g_1}.$$

It should be emphasized that expansions analogous to (32)–(36) will hold along the other simply supported boundary line $x_2 = 0$, $0 < x_1 \leq 1/2$, since the differential equation (26) is symmetrical in the spatial variables x_1 and x_2.

Near the corner of the plate domain, where the two simply supported boundaries intersect, i.e. for small values of x_1 and x_2, we find the following leading terms in a two dimensional series expansion for w, satisfying the differential equation (19) and the boundary conditions (20) and (21),

$$w(x_\alpha) = ax_1x_2(x_1{}^2 + x_2{}^2)^{-1/2} + b(x_1x_2)^{3/2}(x_1{}^2 + x_2{}^2)^{-1} + \dots$$
$$= ar\cos\theta\sin\theta + br(\cos\theta\sin\theta)^{3/2} + \dots . \quad (37)$$

In (37), a and b are constants, and $r = (x_1{}^2 + x_2{}^2)^{1/2}$ and $\theta = \mathrm{Arc}\tan(x_2/x_1)$ denote polar co-ordinates. The agreement between (37) and the expansion (32), which is valid for small values of x_1 along x_2 = constant, is obvious.

From (37), we easily find the leading terms in expansions of Δw, D, h and M, respectively, to be

$$\Delta w = \tfrac{3}{4}b(x_1x_2)^{-1/2} + \dots = \tfrac{3}{4}br^{-1}(\cos\theta\sin\theta)^{-1/2} + \dots, \quad (38)$$

$$D = \frac{64}{27}\frac{A^{3/2}}{|b|^3}(x_1x_2)^{3/2} + \dots = \frac{64}{27}\frac{A^{3/2}}{|b|^3}r^3(\cos\theta\sin\theta)^{3/2} + \dots, \quad (39)$$

$$h \sim D^{1/3} = \frac{4}{3}\frac{A^{1/2}}{|b|}(x_1x_2)^{1/2} + \dots = \frac{4}{3}\frac{A^{1/2}}{|b|}r(\cos\theta\sin\theta)^{1/2} + \dots, \quad (40)$$

$$M = \frac{16}{9}\frac{A^{3/2}}{|b|^3}b\cdot x_1x_2 + \dots = \frac{16}{9}\frac{A^{3/2}}{|b|^3}br^2\cos\theta\sin\theta + \dots . \quad (41)$$

Note that (37) indicates that the coefficient function $g_1(x_2)$ of the singular term in the x_1 expansion (33) for Δw is itself singular at $x_2 = 0$.

SOLUTION BY SUCCESSIVE ITERATIONS

The non-linear eigenvalue problem consisting of the partial differential equation (26), the integral equation (27), and the boundary conditions (28) and (30), can be solved numerically by successive iterations.

The method applied is based upon a formal integration of the fourth order equation (26) in two steps, each of which involves solution of a second order partial differential equation.

Before going into detail on this, it is worth noting that the statical boundary conditions of the problem are most conveniently expressed in terms of the scalar moment function $M(x_\alpha)$, see (30). To invoke these boundary conditions in a simple manner, it is therefore advantageous in applying of (29), to express the differential equation (26) in terms of M and w rather than in terms of D and w.

Using (29), equation (26) takes the following form, where terms containing second order derivatives of M are separated on the left-hand side,

$$\left\{\frac{w_{,11}}{\Delta w} + \nu \frac{w_{,22}}{\Delta w}\right\} M_{,11} + 2(1-\nu)\frac{w_{,12}}{\Delta w} M_{,12} + \left\{\nu \frac{w_{,11}}{\Delta w} + \frac{w_{,22}}{\Delta w}\right\} M_{,22}$$

$$= 3D^{1/3}w - 2(1-\nu)\left\{\frac{\Delta_{,1}w}{\Delta w}\frac{w_{,22}}{\Delta w} - \frac{\Delta_{,2}w}{\Delta w}\frac{w_{,12}}{\Delta w}\right\} M_{,1}$$

$$- 2(1-\nu)\left\{\frac{\Delta_{,2}w}{\Delta w}\frac{w_{,11}}{\Delta w} - \frac{\Delta_{,1}w}{\Delta w}\frac{w_{,12}}{\Delta w}\right\} M_{,2}$$

$$- (1-\nu)\left\{\left(\frac{\Delta_{,11}w}{\Delta w} - 2\left(\frac{\Delta_{,1}w}{\Delta w}\right)^2\right)\frac{w_{,22}}{\Delta w}\right.$$

$$- 2\left(\frac{\Delta_{,12}w}{\Delta w} - 2\frac{\Delta_{,1}w}{\Delta w}\frac{\Delta_{,2}w}{\Delta w}\right)\frac{w_{,12}}{\Delta w}$$

$$\left. + \left(\frac{\Delta_{,22}w}{\Delta w} - 2\left(\frac{\Delta_{,2}w}{\Delta w}\right)^2\right)\frac{w_{,11}}{\Delta w}\right\} M. \tag{42}$$

Since we have not replaced $D^{1/3}$ in the first term on the right-hand side, (42) can be regarded as a linear, second order partial differential equation with variable coefficients for the scalar function M. For this equation, we have the boundary conditions (30).

Solving equation (29) with respect to the highest power of Δw, we obtain the equation

$$\Delta w = \left[(A + w^2)\left[\frac{\Delta w}{M}\right]^{2/3} - 2(1-\nu)[(w_{,12})^2 - w_{,11}w_{,22}]\right]^{1/2}, \tag{43}$$

which, in connection with the boundary conditions (28), constitutes a determining Poisson boundary value problem for the function w, if we assume that the square root function on the right-hand side is known.

Equations (42) and (43) for solving formally M and w, respectively, and the implicit integral condition (27) for the eigenvalue A, constitute the basic steps in the subsequent scheme for successive iterations, where the subscript n refers to the iteration number:

(I) Solve $w_n(x_\alpha)$ from the boundary value problem

$$\begin{cases} \text{P.D.E.:} & \Delta w_n = f_n(x_\alpha) \\ \text{B.C.'s:} & w_n(x_1, 0) = w_n(0, x_2) = 0, \\ & w_{n,1}(1/2, x_2) = w_{n,2}(x_1, c/2) = 0. \end{cases}$$

Hereby, $w_{n,11}$, $w_{n,22}$ and $w_{n,12}$ are also determined.

(II) Inner iteration loop (local subscript i, $i = 1, 2, \ldots$) for determining A_n and $D_n^{1/3}(x_\alpha)$:

(a) $$D_i^{1/3}(x_\alpha) = \left[\frac{A_i + w_n^2}{(\Delta w_n)^2 + 2(1-\nu)[(w_{n,12})^2 - w_{n,11}w_{n,22}]}\right]^{1/2}$$

(b) $$A_{i+1} = 2\,\frac{\int_0^{1/2}\int_0^{c/2} D_i^{1/3} w_n^2 \, dx_1\, dx_2}{\int_0^{1/2}\int_0^{c/2} D_i^{1/3} \, dx_1\, dx_2}.$$

When convergence is obtained, put $A_n = A_i$ and $D_n^{1/3}(x_\alpha) = D_i^{1/3}(x_\alpha)$.

(III) Compute the coefficient functions

$$a_n(x_\alpha) = \frac{w_{n,11}}{\Delta w_n} + v\frac{w_{n,22}}{\Delta w_n}$$

$$2b_n(x_\alpha) = 2(1-v)\frac{w_{n,12}}{\Delta w_n}$$

$$c_n(x_\alpha) = v\frac{w_{n,11}}{\Delta w_n} + \frac{w_{n,22}}{\Delta w_n}.$$

Differentiate Δw_n and compute

$$p_n(x_\alpha) = 2(1-v)\left\{\frac{\Delta_{,1} w_n}{\Delta w_n}\frac{w_{n,22}}{\Delta w_n} - \frac{\Delta_{,2} w_n}{\Delta w_n}\frac{w_{n,12}}{\Delta w_n}\right\}$$

$$g_n(x_\alpha) = 2(1-v)\left\{\frac{\Delta_{,2} w_n}{\Delta w_n}\frac{w_{n,11}}{\Delta w_n} - \frac{\Delta_{,1} w_n}{\Delta w_n}\frac{w_{n,12}}{\Delta w_n}\right\}$$

$$\begin{aligned} r_n(x_\alpha) = (1-v)\Bigg\{&\left[\frac{\Delta_{,11} w_n}{\Delta w_n} - 2\left[\frac{\Delta_{,1} w_n}{\Delta w_n}\right]^2\right]\frac{w_{n,22}}{\Delta w_n} \\ &-2\left[\frac{\Delta_{,12} w_n}{\Delta w_n} - 2\frac{\Delta_{,1} w_n}{\Delta w_n}\frac{\Delta_{,2} w_n}{\Delta w_n}\right]\frac{w_{n,12}}{\Delta w_n} \\ &+\left[\frac{\Delta_{,22} w_n}{\Delta w_n} - 2\left[\frac{\Delta_{,2} w_n}{\Delta w_n}\right]^2\right]\frac{w_{n,11}}{\Delta w_n}\Bigg\}. \end{aligned}$$

(IV) Solve $M_n(x_\alpha)$ from the boundary value problem

$$\begin{cases} \text{P.D.E.:} & a_n(x_\alpha)M_{n,11} + 2b_n(x_\alpha)M_{n,12} + c_n(x_\alpha)M_{n,22} \\ & \quad = 3D_n^{1/3}w_n - p_n(x_\alpha)M_{n-1,1} - g_n(x_\alpha)M_{n-1,2} - r_n(x_\alpha)M_{n-1} \\ \text{B.C.'s:} & M_n(x_1, 0) = M_n(0, x_2) = 0, \\ & M_{n,1}(1/2, x_2) = M_{n,2}(x_1, c/2) = 0. \end{cases}$$

(V) Compute

$$f_{n+1}(x_\alpha) = \left[(A_n + w_n^2)\left[\frac{\Delta w_n}{M_n}\right]^{2/3} - 2(1-v)[w_{n,12}^2 - w_{n,11}w_{n,22}]\right]^{1/2}.$$

The starting point of each iteration is the function $f_n(x_\alpha)$, obtained from the previous iteration. Knowing this function we obtain w_n and its derivatives up to second order by solving the Poisson boundary value problem (I). Now, a value of A_n corresponding to w_n can be determined by the inner successive iteration loop (II). Hereby, the iterate $D_n(x_\alpha)$ is also determined.

In the third step (III), the coefficient functions for M and its derivatives in equation (42) are computed. Note that some differentiations of Δw_n cannot be avoided here. The coefficient functions are used in step (IV), where the new moment function $M_n(x_\alpha)$ is solved from the second order terms on the left-hand side of the differential equation, whereas the moment function and its first derivatives, on the right-hand side, are taken from the previous iteration.

Finally, at the last step (V), we obtain the starting function $f_{n+1}(x_\alpha)$ for the next iteration.

The first iteration is started by means of an arbitrary function $f_1(x_\alpha)$, and by taking the function $M_0(x_\alpha)$ identically zero. We thereby avoid differentiations of $\Delta w_1 = f_1(x_\alpha)$ during the first iteration, in which this function is not yet adapted to higher order boundary conditions.

Though fourth order derivatives of w are in fact found on both sides of equation (42) (M contains second derivatives of w), and though second order derivatives are found on both sides of equation (43), the terms written in the left hand sides† of these equations prove to be sufficiently *dominant* to ensure global convergence by the iteration process.

When convergence is obtained, i.e. when the sequence of iterates has become stationary, the optimal eigenvalue and the corresponding thickness function are computed by means of (24) and (25).

The numerical solution of the problem is based upon finite difference formulations of the equations, and performed by representing discretely the iterates in a rectilinear grid with sides parallel to the coordinate axes.

However, as the solution w has singular second and higher order derivatives normal to the simply supported edges of the plate, it is necessary to determine the type of this singular behaviour analytically, as in the previous section. When this behaviour is known, the singularity of each singular iterate is separated as a multiplying mathematical expression, which is taken care of analytically by specially prepared finite difference approximations and integration procedures. This leaves only the regular remainder of each singular iterate for numerical treatment.

RESULTS

The optimal thickness function for a simply supported *square* plate is illustrated in Fig. 2. Note that the thickness function is proportional to the square root of the normal distance from the edges in the vicinity of the edges, in accordance with the results of the singularity investigation. In the vicinity of the plate corners, the thickness function is directly proportional to the distance from the corner point.

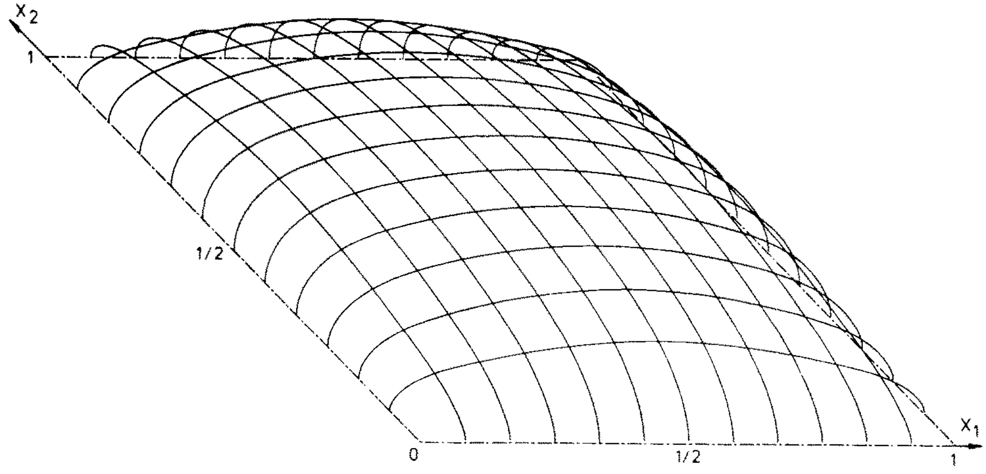

Fig. 2. The optimal thickness function for a simply supported square plate.

† The iterates M_n and w_n, respectively, are solved by integration of these terms.

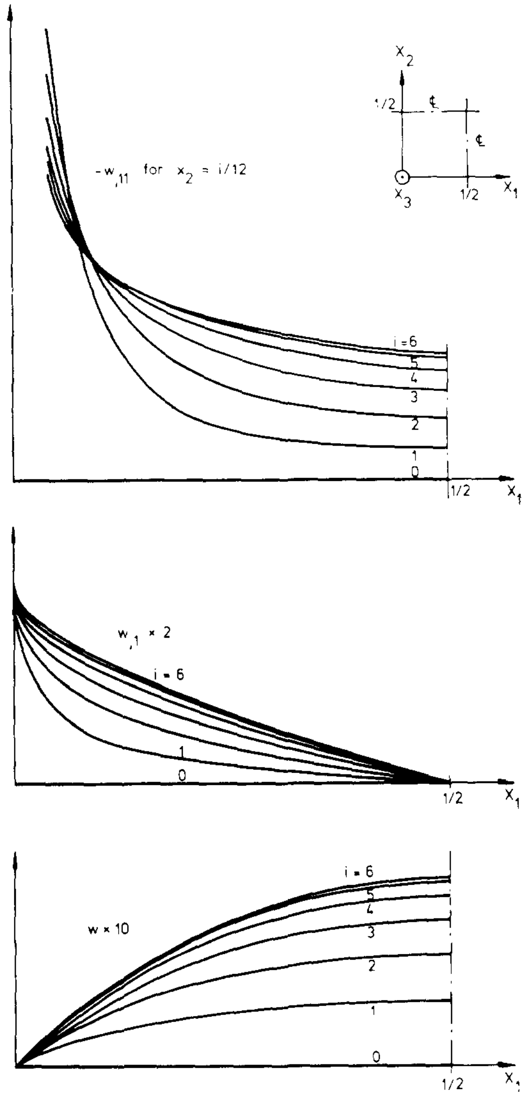

Fig. 3. The deflection function w and some derivatives for the optimal, square plate. The ordinate scales are equal.

For this square plate, Fig. 3 shows the eigenfunction w and its derivatives $w_{,1}$ and $w_{,11}$ for different values of the coordinate x_2. As indicated by the series expansion (32), the second derivative has a square root singularity, while the first derivative is finite along the simply supported x_2-axis. Both functions jump to zero at the corner of the plate.

Figure 4 illustrates the thickness function of an optimal simply supported rectangular plate with the aspect ratio c equal to 2. In Fig. 5, the second derivatives $w_{,22}$ and $w_{,11}$ of the corresponding eigenfunction are shown for different values of the coordinates x_1 and x_2, respectively.

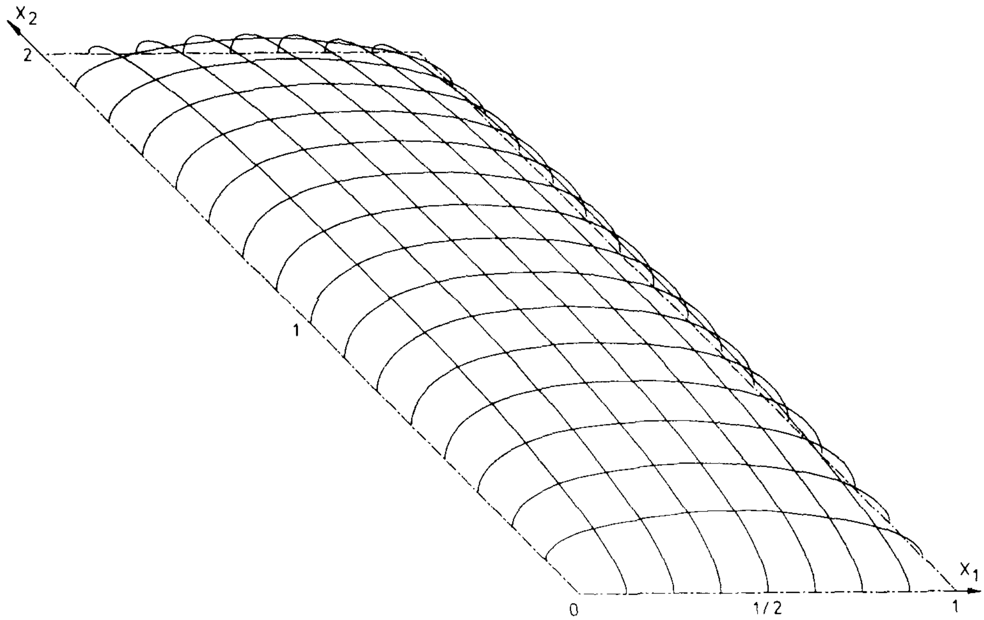

Fig. 4. The optimal thickness function for a simply supported rectangular plate with the aspect ratio $c = 2$.

Beside the singular behaviour, the smallness of $w_{,22}$ near the $x_2 = 1$ plane of symmetry should also be emphasized. This indicates that the eigenfunction w is rather independent of the x_2-coordinate in a sub-domain near the plane of symmetry mentioned. The thickness function too, has this behaviour, see Fig. 4.

In fact, by increasing the aspect ratio c of the plate, we find the x_2-independence in an increasing part of the plate domain, whereas the x_2-dependence remains limited within two small sub-domains close to the shorter sides of the plate.

In Table 1, the optimum fundamental eigenvalue

$$\lambda = \omega^2 \rho a^8 12(1 - \nu^2)/EV^2$$

Table 1.

c	λ	$\omega/\omega_u = \sqrt{(\lambda/\lambda_u)}$
1	468·1	1·096
7/6	261·3	1·102
3/2	115·0	1·128
2	51·73	1·166
5/2	29·44	1·185
3	18·97	1·192
7/2	13·22	1·192
4	9·740	1·190
5	5·908	1·184
6	3·962	1·177
7	2·844	1·172
10	1·346	1·164
15	0·5866	1·159

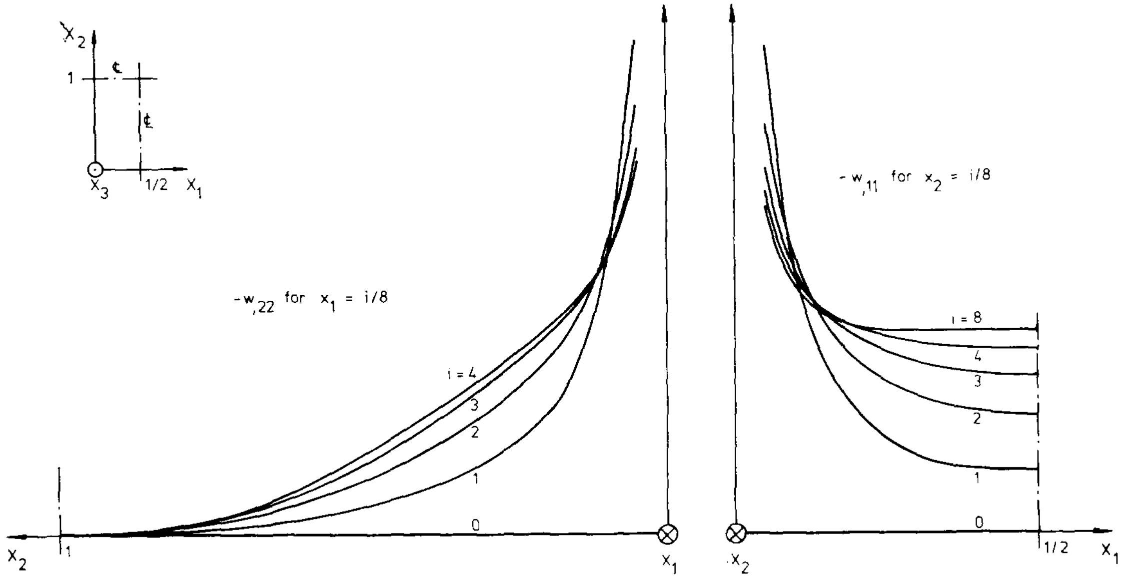

Fig. 5. Second derivatives of the deflection function w for an optimal rectangular plate with the aspect ratio $c = 2$.

for a simply supported plate is listed for different values of the aspect ratio c. The right-hand column in the Table gives the ratio between the optimum fundamental angular frequency ω and the corresponding frequency ω_u of a *uniform* plate of equal boundary, volume and material.

The results presented in this section presuppose Poisson's ratio to be equal to 0·3. It should be emphasized that the governing differential equation and the boundary conditions contain ν, which will influence the shape and the eigenvalue λ of the optimal plate. This influence was found to be weak. Young's modulus E and the mass density ρ affect the natural angular frequency ω, only.

Asymptotical behaviour for $c \to \infty$

It can be shown that except for a scaling factor, we as a special case obtain the optimal height function of a simply supported vibrating solid *beam* of constant width in the x_2-independent domain of a sufficiently *long* plate.

In fact, by cancelling all x_2-derivatives in equations (16), (18) and (19) and neglecting the deviations from x_2-independence in the integrands of (18), the double integrals of this equation can be reduced to single integrals with the aspect ratio c as a multiplying factor. Thereby, the governing equations for w, A and h of the plate optimization problem transform exactly into the corresponding *ordinary* differential equations for the solid beam optimization problem. The only exception is the equation for the optimal beam eigenvalue, which cannot be obtained in this way.

Assuming that the x_2-independence of the solutions hold for an infinitely long plate, it can be shown that the plate frequency ratio ω/ω_u mentioned above has an asymptotical value, as c tends to infinity. This value is equal to the corresponding frequency ratio for simply supported beams of constant width and, as a special case in [5], is found to be 1·119.

FINAL REMARKS

The optimality equations derived for solid plates in this paper are necessary conditions only, and they may have a variety of solutions.

The numerical solutions obtained have rather smooth thickness variations, and they must be considered as *local optima*, since the fundamental frequency would be further increased by using some of the given plate volume to introduce integral stiffeners.

Solutions to the present optimality equations, having *stiffener-like* thickness variations, are likely to exist. However, considerable thickness variations will inevitably imply changes in the sign of the scalar moment function along interior curves in the plate domain. The vanishing of the moment function along these curves will result in zero thickness and singularities in the second derivatives of the deflection at each curve. Except for a finite deflection, the behaviour will be similar to that near the simply supported edges.

However, the possibility of seeking and obtaining solutions that are also singular along internal curves in the plate domain has not been implemented in the present numerical technique. As a matter of fact, solutions of this type will also be local optima, since there *is not a global optimum* solution. This fact can be explained as follows.

Consider first a given, simply supported rectangular plate of size $a \times ac$ and volume V, see Fig. 6a. Now, let us apply, say, half of this volume to form two stiffeners with rectangular

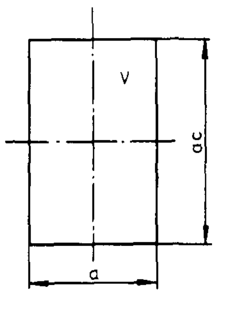

Fig. 6a. Original plate.

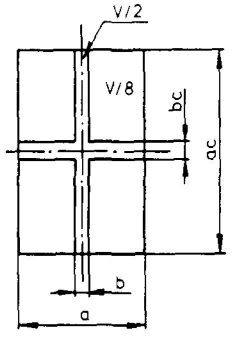

Fig. 6b. Plate structure composed by stiffeners and plate elements.

cross sections of width b and bc, respectively, along the planes of symmetry as shown in Fig. 6b. Furthermore, let us distribute the remainder of the volume in four identical plate elements, geometrically similar to the original plate. As illustrated in Fig. 6, we let the resulting plate structure retain the side lengths a and ac of the original plate.

By making the stiffeners sufficiently high (thereby reducing the widths), we may increase their bending stiffness so that they can provide almost ideal simple supports for the plate elements. Furthermore, we achieve that the fundamental frequency of transverse vibrations of the resulting plate structure will equal the frequency of the plate elements. Now, using (2) for physically and geometrically similar plates, we easily find that this frequency is at least twice that of the original plate.

Performing a similar process for each of the new plate elements in the structure, we at least double the frequency once more, and so forth.

Thus, using some of the prescribed plate volume to form sufficiently many, very high and very thin stiffeners, we can exceed any finite value for the fundamental frequency. This indicates that a global optimum solution does not exist in the present sense.

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